Fast Calculation of Parameter Dependencies in Dielectric Waveguides Using Sensitivity Analysis

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***Abstract*— We perform a sensitivity analysis of the eigenmodes in dielectric waveguides with respect to design parameters. Based on a discretization using the Finite Integration Technique the eigenvalue problem for the wave number is shown to be non-Hermitian with possibly complex solutions even in the lossless case. Nevertheless, the sensitivity can be obtained with negligible numerical effort. The first numerical example, the sensitivity of the effective index of a graded index fiber with respect to the core size, demonstrates the validity of the method. For the frequency itself as parameter, a 2nd order sensitivity analysis yields a fast approximation of the dispersion relation of the fiber.**

I. Introduction

Numerical simulations of waveguide structures by finite methods have been used for many years, and a number of eigenvalue formulations are available. For optical applications such as fibers or integrated waveguides the number of unknowns can become quite large although only two-dimensional discrete models are considered. If additionally the dependencies of the modes w.r.t. design parameters such as geometric dimensions or material parameters are searched for, it is desirable to have sophisticated approaches for fast parameter sweeps at hand.

Several such approaches have been reported in the context of Model Order Reduction (MOR) techniques. In most cases they are based on projections of the system matrices by low-dimensional subspaces, and recently some effort has been taken to extend them to the multi-variate case [1], [2].

A different approach, the so-called sensitivity analysis of electromagnetic systems using adjoint techniques [3], has recently gained large interest. Starting with analytical differentiations of the algebraic matrix equations, compact formulas can be derived for the sensitivities of output quantities w.r.t. an arbitrary number of design parameters. Adjoint techniques have been applied to various formulations in electromagnetic modeling, but to our knowledge not yet to eigenvalue problems of dielectric waveguides. In this paper we apply a classical sensitivity analysis to the eigenvalue problem arising from a 2D FIT-discretization of inhomogeneous dielectric waveguides.

II. Waveguide eigenvalue problem using FIT

We consider a cross section of a dielectric waveguide and use the Finite Integration Technique, FIT, [4], [5] for the discretization of Maxwell's Equations in frequency domain. For sake of simplicity, a standard Cartesian mesh with PEC boundary conditions is used. The derivation of the resulting eigenvalue problem has been described in detail in [4], [6] and is only briefly revisited here.

The state variables of FIT are integral quantities which are defined on edges and facets of the primary grid G and the dual grid $\widetilde{G}$, respectively. Collected in algebraic vectors, these are the grid voltages $\widehat{\mathbf{e}}$, $\widehat{\mathbf{h}}$ and the grid fluxes $\widehat{\widehat{\mathbf{d}}}$, $\widehat{\widehat{\mathbf{b}}}$. Neglecting charges and currents, we obtain the *Maxwell's Grid Equations*

$$\mathbf{C}\,\widehat{\mathbf{e}} = -j\omega\,\widehat{\widehat{\mathbf{b}}}, \qquad \widetilde{\mathbf{C}}\,\widehat{\mathbf{h}} = j\omega\,\widehat{\widehat{\mathbf{d}}}, \tag{1}$$

$$\mathbf{S}\,\widehat{\widehat{\mathbf{b}}} = \mathbf{0}, \qquad \widetilde{\mathbf{S}}\,\widehat{\widehat{\mathbf{d}}} = \mathbf{0}. \tag{2}$$

The matrices $\mathbf{C}$ and $\widetilde{\mathbf{C}} = \mathbf{C}^T$ are the discrete curl-operators, matrices $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ the discrete div-operators of the primary and dual grid, respectively. In Cartesian grids they consist of submatrices $\mathbf{P}_x$, $\mathbf{P}_y$, $\mathbf{P}_z$ which can be identified as partial differentiation operators [5]. From grid topology we find the exact relations $\mathbf{SC} = \mathbf{0}$ and $\widetilde{\mathbf{S}}\widetilde{\mathbf{C}} = \mathbf{0}$.

The formulation is completed by the *material relations* (for linear media) $\widehat{\widehat{\mathbf{d}}} = \mathbf{M}_\varepsilon\,\widehat{\mathbf{e}}$, $\widehat{\mathbf{h}} = \mathbf{M}_\mu^{-1}\,\widehat{\widehat{\mathbf{b}}}$. Both material matrices $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\mu^{-1}$ are diagonal for Cartesian meshes and may be complex to account for dielectric or magnetic losses.

For the discretization of waveguide cross sections we use a 2D Cartesian grid system with N_P primary nodes. If we assume a wave propagation $\vec{E}, \vec{H} \sim e^{-j\,k_z z}$ in z-direction with the wave number k_z, the longitudinal differentiation operator is given by $\mathbf{P}_z = -jk_z\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix.

In order to derive an eigenvalue formulation for the modes in such waveguide cross sections, we start with the 3D curl-curl eigenproblem $\mathbf{C}^T\mathbf{M}_\mu^{-1}\mathbf{C}\,\widehat{\mathbf{e}} = \omega^2\mathbf{M}_\varepsilon\,\widehat{\mathbf{e}}$ and use the divergence-free condition of the fields, $\widetilde{\mathbf{S}}\,\widehat{\widehat{\mathbf{d}}} = \mathbf{0}$, to eliminate the longitudinal $\widehat{\mathbf{e}}_z$-components. This leads to a $2N_P \times 2N_P$-eigenvalue problem for the transversal electric field:

$$(\mathbf{A}_{CC} - \omega^2\mathbf{B} + k_z^2\mathbf{I})\,\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} = \begin{pmatrix} \widehat{\mathbf{e}}_x \\ \widehat{\mathbf{e}}_y \end{pmatrix}. \tag{3}$$

For a fixed frequency ω this is a simple, non-symmetric eigenproblem $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$ with the eigenvalue $\lambda = -k_z^2$.

III. Sensitivity analysis

We are interested in the sensitivity of the eigensolutions with respect to a number of design parameters such as geometric dimensions or permittivity values. For simplicity of notation we restrict here to one single parameter p and calculate the derivatives $\lambda' = \mathrm{d}\lambda/\mathrm{d}p$ and $\mathbf{x}' = \mathrm{d}\mathbf{x}/\mathrm{d}p$.

The derivation makes use of the left-eigenvectors $\mathbf{y}$ of the system (the eigenvector of the Hermitian matrix $\mathbf{A}^H$, hence: *adjoint technique*) with

$$\mathbf{y}^H(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0} \tag{4}$$

and the orthonormality condition of right- and left-eigenvectors (of two modes i, j)

$$\mathbf{y}^{(j)H}\mathbf{x}^{(i)} = \delta_{ij}. \tag{5}$$

Following the standard perturbation theory for eigenproblems [7] we build the derivative $\frac{\mathrm{d}}{\mathrm{d}p}\{(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}\}$:

$$\Rightarrow \quad (\mathbf{A}'-\lambda'\mathbf{I})\mathbf{x} + (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}' = \mathbf{0}. \tag{6}$$

Multiplying from the left by $\mathbf{y}^H$ yields, together with (4) and (5), the desired eigenvalue sensitivity:

$$\lambda' = \mathbf{y}^H\mathbf{A}'\mathbf{x}. \tag{7}$$

Once λ' has been calculated, eq. (6) defines a linear system for the eigenvector derivative $\mathbf{x}'$, cf. [7]. In a similar manner, we can also find formulas for higher order derivatives. An example which will be used below is the expression

$$\lambda'' = \mathbf{y}^H\mathbf{A}''\mathbf{x} + 2\mathbf{y}^H(\mathbf{A}'-\lambda'\mathbf{I})\,\mathbf{x}'. \tag{8}$$

Unfortunately, in a non-Hermitian system as given here, the left- and right eigenvectors are not identical. However, the orthogonality property (5) between the left eigenvector and the original right eigenvector (the transversal electric field) suggests that $\mathbf{y}$ may be related to the magnetic field in the guide and the $\vec{E}\times\vec{H}$ cross product. It has already been shown previously [6] that this type of orthogonality can be reproduced within the discrete setting by

$$\sum_n \left(\widehat{e}^{\,(i)}_{x,n}\,\widehat{h}^{(j)}_{y,n} - \widehat{e}^{\,(i)}_{y,n}\,\widehat{h}^{(j)}_{x,n}\right) = \delta_{ij} \quad \Rightarrow \mathbf{y} = \begin{pmatrix} \widehat{\mathbf{h}}^*_y \\ -\widehat{\mathbf{h}}^*_x \end{pmatrix} \tag{9}$$

To confirm this assumption we can derive the eigenmode formulation for the magnetic field components in a similar way than above for the electric field. After some calculation we indeed find that $\mathbf{y}$ solves the adjoint eigenvalue problem to (3). As a consequence, although the matrix is non-symmetric, the left-eigenvector can be easily calculated without any additional solver step, simply by applying the discrete Faraday's law.

Finally we need an implementation for the matrix derivative $\mathbf{A}' = \frac{\partial}{\partial p}\mathbf{A}$, more details will be given in the final paper.

IV. Numerical Example

We test our algorithm on a graded index waveguide with parabolic index profile. Fig. 1 shows the 2D computational grid and the profile of the refractive index $n = \sqrt{\varepsilon_r}$ as a function of the radius ρ. Due to the twofold symmetry only a quarter of the waveguide has to be discretized. The goal quantity is the effective refractive index $n_{\text{eff}} = k_z/k_0$, and parameter sweeps serve as a reference for the sensitivities.

The first parameter in the sensitivity analysis is the core radius a of the waveguide. Fig. 2 shows the reference results together with a tangent which uses the first order derivative from the sensitivity analysis. The results fit very nicely.

In a second validation we calculate the dispersion relation of the guide, i.e. the dependency of the effective refraction index on the frequency f. From (3) it is obvious that the matrix derivative (w.r.t. to ω^2) is simply $\mathbf{A}' = -\mathbf{B}$ in this case. Compared again to a parameter sweep as reference, the 2nd plot in Fig. 2 shows the relative deviation of a first and second order approximation, using the first and second derivative at a single expansion point, respectively.

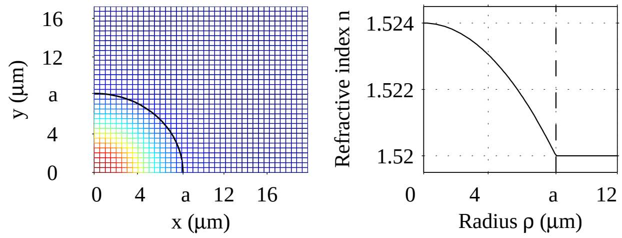

Fig. 1. Geometrical data of the graded index waveguide profil.

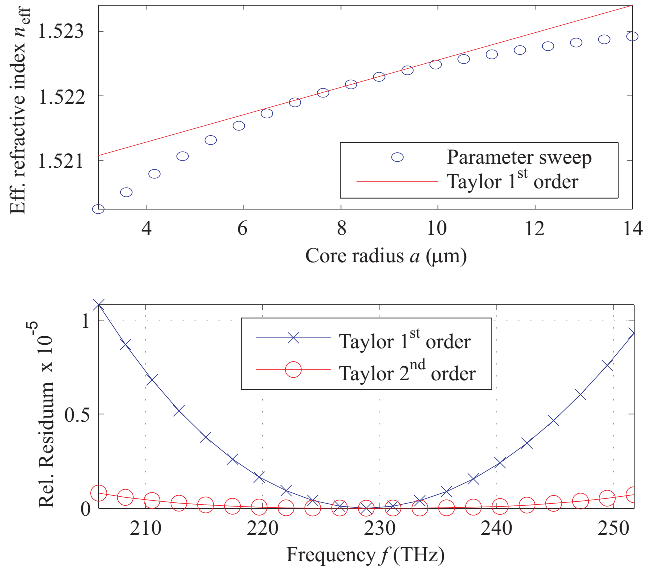

Fig. 2. Sensitivity analysis of $n_{\text{eff}}(a)$ and $n_{\text{eff}}(f)$.

V. Conclusions

Since the required left-eigenvectors in the sensitivity analysis are available without an additional solving step, the first and second order derivatives w.r.t. various parameters can be calculated at low computational cost. The results may be useful in optimization approaches, but also as a fast alternative to parameter sweeps. Applied to the frequency as parameter, this approach clearly has close relations to MOR techniques which will be further discussed in the final paper.

References

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